UDC 533.6.011

HYPERSONIC AXISYMMETRIC FLOW WITH CONSTANT ANGULAR MOMENTUM

PMM Vol.41, № 1, 1977, pp.65-71

E. D. TERENT'EV
(Moscow)

(Received June 11, 1976)

An axisymmetric hypersonic flow at large distances from the streamlined body is considered. Behind the body the flow splits into three regions: the region containing the shock wave structure, the external ideal flow, and the laminar viscous and heat conducting wake. Perturbations associated with the constant moment of momentum directed along the axis of symmetry, are studied. The perturbations constructed are localized in the wake and decay exponentially on passing to the outer region.

We shall consider an axisymmetric steady state flow of gas. We denote by ρ_∞ the gas density in the incoming stream moving at velocity v_∞ which is collinear with the x-axis of the cylindrical x, r, θ coordinate system. Neglecting the pressure of gas in the incoming stream we set $p_\infty = 0$, from this it follows that the Mach number $M_\infty = \infty$. We assume the gas to be perfect with both specific heat capacities c_p and c_v constant. We denote the Prandtl number by N_{Pr} and assume for simplicity that the coefficients of viscosity λ_1 and λ_2, and of heat conductivity k, are linearly dependent on the specific enthalpy w, namely $\lambda_1 = \lambda_{10}w$, $\lambda_2 = \lambda_{20}w$, $k = k_0 w$. In what follows it is convenient to assume that the independent variables and the unknown functions are dimensionless, and use ρ_∞, v_∞ and λ_{10} as the basic units of reference. We use the system of Navier-Stokes equations written in the dimensionless form, as the basic system

$$\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_r}{\partial r} + \frac{\rho v_r}{r} = 0, \quad p = \frac{\varkappa - 1}{\varkappa}\rho w \quad \left(\varkappa = \frac{c_p}{c_v}\right) \tag{1}$$

$$\rho v_x \frac{\partial v_x}{\partial x} + \rho v_r \frac{\partial v_x}{\partial r} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left\{w\left[\frac{4}{3}\frac{\partial v_x}{\partial x} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right) + \frac{\lambda_{20}}{\lambda_{10}}\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_x}{\partial x}\right)\right]\right\} + \frac{\partial}{\partial r}\left[w\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)\right] + \frac{w}{r}\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)$$

$$\rho v_x \frac{\partial v_r}{\partial x} + \rho v_r \frac{\partial v_r}{\partial r} - \rho\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial x}\left[w\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)\right] + \frac{\partial}{\partial r}\left\{w\left[\frac{2}{3}\left(2\frac{\partial v_r}{\partial r} - \frac{v_r}{r} - \frac{\partial v_x}{\partial x}\right) + \frac{\lambda_{20}}{\lambda_{10}}\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{s} + \frac{\partial v_x}{\partial x}\right)\right]\right\} + \frac{2w}{r}\left(\frac{\partial v_r}{\partial r} - \frac{v_r}{r}\right)$$

$$\rho v_x \frac{\partial v_\theta}{\partial x} + \rho v_r \frac{\partial v_\theta}{\partial r} + \rho\frac{v_r v_\theta}{r} = \frac{\partial}{\partial x}\left(w\frac{\partial v_\theta}{\partial x}\right) + \frac{\partial}{\partial r}\left[w\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)\right] + \frac{2w}{r}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

$$\rho v_x \frac{\partial w}{\partial x} + \rho v_r \frac{\partial w}{\partial r} = v_x \frac{\partial p}{\partial x} + v_r \frac{\partial p}{\partial r} + \frac{\partial}{\partial x}\left(\frac{1}{N_{\mathrm{Pr}}} w \frac{\partial w}{\partial x}\right) +$$
$$\frac{\partial}{\partial r}\left(\frac{w}{N_{\mathrm{Pr}}}\frac{\partial w}{\partial r}\right) + \frac{1}{r}\frac{1}{N_{\mathrm{Pr}}}\frac{\partial w}{\partial r} + 2w\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{v_r}{r}\right)^2 +\right.$$
$$\left.\frac{1}{2}\left(\frac{\partial x_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v_\theta}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2\right] +$$
$$\left(-\frac{2}{3} + \frac{\lambda_{20}}{\lambda_{10}}\right) w \left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_x}{\partial x}\right)^2$$

It was established in [1] that in the case of flows with infinitely large Mach number the uniform incoming flow is separated from the nonuniform one by a line of discontinuity in the derivatives of the gas dynamic functions. This line forms a boundary of the region of shock wave structure. The decisive part in the formation of flow in this region is played by the normal viscous stresses and the normal component of the thermal flux vector. We use the method of matching the outer and inner asymptotic expansions [2] to pass from the region of the shock wave structure to the outer region, in which the influences of viscosity and heat conductivity become insignificant in the first approximation. For a flow in the outer region an analogy [3 — 6] holds, which compares the hypersonic flow with an unsteady flow in a space of dimension less by one. Using this analogy we can find the parameters of the flow behind a finite body from the solution of the problem of strong explosion [7 — 9]. The solution of the problem of strong explosion can, in turn, be extended on approaching the axis of the flow, to the region of laminar wake [10] where the tangential component of the heat flux vector and the tangential viscous stresses play a decisive role in the formation of the flow.

In the present paper we consider a hypersonic flow corresponding to a flow past a body in which the moment of momentum M_x directed along the symmetry axis is conserved. The reasons for the constancy of the moment of momentum are not essential in the first approximation. Since the gas is assumed viscous, this could be caused by a twisted stream or by a uniformly rotating body. To calculate M_x we introduce two control planes perpendicular to the flow axis. Let one of them be situated ahead the body and the other, denoted by Σ, behind the body at the distance x. Remembering that the incoming flow is uniform, we have the following expression for the dimensionless moment:

$$M_x = -\iint\limits_{\Sigma} \left(\rho v_x v_\theta - w \frac{\partial v_\theta}{\partial x}\right) r^2 d\theta\, dr \tag{2}$$

Clearly, the quantity M_x should not be dependent on the distance x at which the control plane Σ is situated.

The authors of [11] proposed a general method for constructing unsteady flows of a perfect gas in which the basic self-similar flow is subjected to perturbations connected with any one physical quantity and maintained over a period of time. The results of [11] were extended in [12, 13] to embrace the stationary hypersonic flows and and flows were constructed behind the bodies subjected not only to drag, but also to lift. The basic idea of this approach consists of the fact that we begin the search for the perturbations by stipulating at once a condition ensuring that some physical quantity is maintained, namely, for the unsteady flows we stipulate that they be independent of time t, or in the case of the steady hypersonic flows — independence of the coordinate x. Naturally, in the case of a complex flow separated into regions such requirement is connected with the hypothesis that integration over one of these regions yields a finite contribution to the physical quantity under consideration. It must also be shown that continuation of perturbations into the remaining regions does not lead to appearance of any singularities.

In the flow under consideration we have three regions: the region of shock wave structure, the outer region and the laminar wake. Let us estimate the moment of momentum M_x for each of these regions. Since the formula (2) defining M_x contains tangential velocity v_θ, we shall verify the feasibility of introducing v_θ in each of the regions.

The region of shock wave structure is separated from the uniform flow by the line of discontinuity in the derivatives of the gas dynamic functions. The line is assumed symmetrical and has the form $r_s = (bx)^{1/2} + \ldots$ when $x \to \infty$. We shall use the estimates given in [2] for the functions within this region. Integrating the projection of the equation of motion on the θ-axis from the system (1), we obtain

$$\rho v_\theta v_r r^2 = r^2 w \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) + f_1(x) \qquad (3)$$

where $f_1(x)$ is an arbitrary function. The boundary conditions at $r = r_s$, where the functions in question have the values $v_\theta = 0$, $v_r = 0$, $w = 0$, require that we set $f_1(x) = 0$. Integration of (3) now yield

$$v_\theta = f_2(x) \exp\left[\int_{r_s}^{r} \left(\frac{1}{r_1} + \frac{\rho v_r}{w}\right) dr_1\right]$$

Using the asymptotics of the functions ρ, v_r and w for $r \to r_s$ given in [2], we can show that the term $\rho v_r / w$ yields an integrable singularity, therefore the fact that $v_\theta = 0$ for $r = r_s$ compels us to set $f_2(x) = 0$. This implies that we cannot introduce perturbations connected with the moment of momentum into the region of shock wave structure, and on the passage to the outer region, we must take $v_\theta = 0$ as the boundary condition. This corresponds to the Rankine-Hugoniot condition of preserving the tangential velocity component during the passage through the shock wave.

Let us now consider the flow in the outer region. The flow can be constructed in the first approximation proceeding from the solution of the problem of strong explosion [7, 8], the latter taken in the form

$$v_x = 1 - \frac{1}{2(\varkappa+1)}\frac{b}{x}U_1(\xi), \quad v_r = \frac{1}{\varkappa+1}\left(\frac{b}{x}\right)^{1/2} V_1(\xi) \qquad (4)$$
$$\rho = \frac{\varkappa+1}{\varkappa-1}\rho_1(\xi), \quad P = \frac{1}{2(\varkappa+1)}\frac{b}{x}P_1(\xi)$$
$$w = \frac{\varkappa}{2(\varkappa+1)^2}\frac{b}{x}W_1(\xi), \quad \xi = \frac{r}{(bx)^{1/2}}$$

We supplement the system (4) with the tangential velocity component

$$v_\theta = \frac{b^{1/2}}{\varkappa+1} x^{-\alpha} Z_1(\xi) \qquad (5)$$

We use the condition that M_x is independent of x to find that $\alpha = 3/2$. Substituting (4) and (5) into (1), we obtain the following corollaries from the equation of continuity and from the projection of the equation of motion on the θ-axis:

$$\left(V_1 - \frac{\varkappa+1}{2}\xi\right)\frac{d\rho_1}{d\xi} + \left(\frac{dV_1}{d\xi} + \frac{V_1}{\xi}\right)\rho_1 = 0 \qquad (6)$$
$$\left(V_1 - \frac{\varkappa+1}{2}\xi\right)\rho_1\frac{dZ_1}{d\xi} + \left(\frac{V_1}{\xi} - \frac{3(\varkappa+1)}{2}\right)\rho_1 Z_1 = 0$$

Multiplying the first equation of (6) by Z_1 and adding the result to the second equation, we obtain

$$\rho_1 Z_1 \xi^2 \left(V_1 - \frac{\varkappa+1}{2}\xi\right) = C_1$$

where C_1 is a constant. We now use the condition at the boundary of the outer flow at $\xi = 1$ where $v_\theta = 0$ and therefore $Z_1(1) = 0$. This at once yields $C_1 = 0$. From this we conclude that in the outer flow $Z_1 \equiv 0$ and a finite contribution towards M_x cannot be obtained.

It remains to consider the region of wake. In the first approximation the flow in the wake region has the form [10]

$$v_x = 1 - \frac{1}{2(\varkappa+1)} b x^{-\varkappa/(\varkappa+1)} U_2(\zeta) + \ldots \tag{7}$$

$$v_r = \frac{1}{\varkappa+1} b^{1/2} x^{-\varkappa/(\varkappa+1)} V_2(\zeta) + \ldots$$

$$\rho = \frac{\varkappa+1}{\varkappa-1} x^{-1/(\varkappa+1)} \rho_2(\zeta) + \ldots$$

$$p = \frac{1}{2(\varkappa+1)} \frac{b}{x} P_2(\zeta) + \ldots$$

$$w = \frac{\varkappa}{2(\varkappa+1)^2} b x^{-\varkappa/(\varkappa+1)} W_2(\zeta) + \ldots, \qquad \zeta = \frac{r}{b^{1/2}} x^{-1/(\varkappa+1)}$$

On approaching the axis of the flow $(\zeta \to 0)$ the functions (7) have the following expansions:

$$U_2(\zeta) = U_{20} + \ldots, \quad V_2 = {}^1/_2\zeta + \ldots \tag{8}$$

$$\rho_2(\zeta) = \rho_{20} + \ldots, \quad P_2 = P_{10}, \quad W_2(\zeta) = W_{20} + \ldots$$

where $U_{20}, \rho_{20}, P_{10}, W_{20}$ are some positive constants. The functions (7) are monotonous. When $\zeta \to \infty$, they have asymptotics which transform into the asymptotics of the solution of the strong explosion problem [14] near its center

$$U_2(\zeta) = U_{10}\zeta^{-2/(\varkappa-1)} + \ldots, \quad V_2 = \frac{\varkappa+1}{2\varkappa}\zeta + \ldots \tag{9}$$

$$\rho_2(\zeta) = \rho_{10}\zeta^{2/(\varkappa-1)} + \ldots, \quad P_2(\zeta) = P_{10}, \quad W_2(\zeta) = W_{10}\zeta^{-2/(\varkappa-1)} + \ldots$$

where $U_{10}, \rho_{10}, P_{10}, W_{10}$ are some positive constants.

Let us supplement the functions (7) with the tangential velocity component

$$v_\theta = \frac{1}{\varkappa+1} b^{1/2} x^{-\beta} Z_2(\zeta) + \ldots \tag{10}$$

Substituting (7) and (10) into (2) we obtain the following expression for the moment of momentum:

$$M_x = -\frac{b^2}{\varkappa-1} x^{2/(\varkappa-1)-\beta} I_0, \quad I_0 = \lim_{\zeta\to\infty} \int_0^\zeta \rho_2 Z_2 \zeta_1^2 d\zeta_1 \tag{11}$$

As before, we choose β from the condition that M_x is independent of the coordinate x, and find that $\beta = 2/(\varkappa+1)$. It is clear that the projection of the equation of motion on the θ-axis in the system (1) could be replaced by the projection of the equation of conservation of the moment of momentum on the x-axis

$$\frac{\partial}{\partial r}\left[r^2 \rho v_\theta v_r - r^2 w \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)\right] + \frac{\partial}{\partial x}\left[r^2 \rho v_x v_\theta - r^2 w \frac{\partial v_\theta}{\partial x}\right] = 0 \tag{12}$$

We note that the integrand in (2) appears also under the derivative with respect to x in

(12). Since the motion in question is related to the constant quantity M_x, the functions (7) and (10) will have the first integral [15]. Performing the manipulations similar to those of [15], we obtain

$$\frac{\varkappa(\varkappa-1)}{2(\varkappa+1)^2}\zeta^2 W_2\left(\frac{dZ_2}{d\zeta}-\frac{Z_2}{\zeta}\right)-\zeta^2 Z_2\rho_2(V_2-\zeta)=C_2 \quad (13)$$

Let us investigate the convergence of the integral I_0 when $\zeta\to 0$ and $\zeta\to\infty$. Using (8), we compute the asymptotics of Z_2 when $\zeta\to 0$. This yields

$$Z_2=-\frac{(\varkappa+1)^2}{\varkappa(\varkappa-1)}\frac{C_2}{W_{20}}\frac{1}{\zeta}+\ldots$$

The above expression implies that although the function $Z_2\to\infty$, nevertheless the integral I_0 converges, when $\zeta\to 0$, for any value of C_2.

Next we inspect the behavior of Z_2 when $\zeta\to\infty$. Using the asymptotics (9) we obtain

$$Z_2=C_2\frac{2\varkappa}{\varkappa-1}\frac{1}{\rho_{10}}\zeta^{-2/(\varkappa-1)-3}+\ldots \quad (14)$$

Naturally, when $\zeta\to\infty$, the asymptotics of the solution of (13) must ensure that the integral I_0 in (11) converges. Substituting (14) into the integral defining I_0, we obtain

$$\int_0^\zeta \rho_2 Z_2\zeta_1^2 d\zeta_1\sim C_2\ln\zeta$$

From this we conclude at once that the finiteness of M_x is equivalent to the requirement that $C_2=0$. Integrating now (13), we obtain

$$Z_2=C_3\zeta\exp\left[\frac{2(\varkappa+1)^2}{\varkappa(\varkappa-1)}I_1(\zeta)\right]$$

$$I_1(\zeta)=\int_0^\zeta\frac{\rho_2}{W_2}(V_2-\zeta_1)\,d\zeta_1$$

when $\zeta\to 0$, the integral $I_1(\zeta)\to 0$, consequently $Z_2\to C_3\zeta$. The behavior of Z_2 when $\zeta\to\infty$ is determined using the asymptotics (9)

$$Z_2=C_3\zeta\exp\left[-\frac{(\varkappa-1)(\varkappa+1)}{2\varkappa^2}\frac{\rho_{10}}{W_{10}}\zeta^{2(\varkappa+1)/(\varkappa-1)}\right] \quad (15)$$

Since the exponential part of the expression (15) is negative, $(\rho_{10}>0, W_{10}>0)$, and tends to infinity as $\zeta\to\infty$, it follows that Z_2 decays exponentially. This implies that the integral I_0 converges. The formula (11) easily yields the relation connecting the constant C_3 with M_x:

$$M_x=-\frac{b^2}{\varkappa-1}C_3 I_2$$

$$I_2=\int_0^\infty\rho_2\exp\left[\frac{2(\varkappa+1)}{\varkappa(\varkappa-1)}I_1(\zeta)\right]\zeta^3 d\zeta$$

The function Z_2 is shown in Fig. 1 for $C_3=1$, $\varkappa=1{,}4$, $N_{\mathrm{Pr}}={}^3/_4$ and the constant $I_2=0.0833$.

Thus the perturbations which are connected by the constant moment of momentum directed along the x-axis, are localized within the wake and decay exponentially on the passage from the wake into the outer flow. This leads us to conclusion that, within the approximation considered, the continuation of v_θ into the outer region will be represented by a function identically equal to zero.

The flow constructed is always rotational, but when $M_x = 0$, the component ω_x of the velocity vortex along the flow axis is identically equal to zero. When M_x appears, the latter emerges and we have

$$\omega_x = \frac{1}{\varkappa+1} x^{-3/(\varkappa+1)}\left(\frac{dZ_2}{d\zeta} + \frac{Z_2}{\zeta}\right)$$

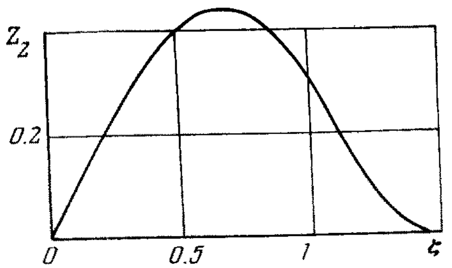

Fig. 1

In accordance with the asymptotic behavior of Z_2, the component ω_x of the vortex has no singularities on the axis and decreases exponentially when $\zeta \to \infty$. However, when the moment of momentum M_x is conserved at $x \to \infty$, ω_x tends to zero. This is explained by the dissipative processes taking place within the gas.

In carrying out the computations, the fact that the longitudinal velocity v_x was different from unity was never taken into account. For this reason an analogy exists for the present problem between the hypersonic and unsteady gas flows. The solutions obtained describe the motion of an initially cold gas in which energy is emitted at the initial instant of time along some axis, and an angular velocity is imparted which ensures the presence of a finite moment of momentum directed along this axis.

REFERENCES

1. Sychev, V. V., On hypersonic flows of viscous heat-conducting gases. PMM Vol. 25, № 4, 1961.
2. Terent'ev, E. D., The structure of shock waves in hypersonic flows. PMM Vol. 38, № 2, 1974.
3. Tsien, H. S., Similarity laws of hypersonic flows. J. Math. and Phys., Vol. 25, № 3, 1946.
4. Hayes, W. D., On hypersonic similitude, Quart. Appl. Math., Vol. 5, № 1, 1947.
5. Il'iushin, A. A., The law of plane sections in the aerodynamics of high supersonic speeds. PMM Vol. 20, № 6, 1956.
6. Bam-Zelikovich, G. M., Bunimovich, A. I. and Mikhailova, M. P., The motion of slender bodies at high supersonic speeds. Izv. Akad. Nauk SSSR, OTN, Mekhanika i Mashinostroenie, № 1, 1960.
7. Sedov, L. I., Motion of air in a strong explosion. Dokl. Akad. Nauk SSSR, Vol. 52, № 1, 1946.
8. Sedov, L. I., Propagation of strong waves of discontinuity. PMM Vol. 10, № 2, 1946.
9. Taylor, G. I., The formation of a blast wave by a very intense explosion, II. The atomic explosion, 1945. Proc. Roy. Soc. A, Vol. 201, № 1065, 1950.

10. Sychev, V. V., Flow in a laminar hypersonic wake behind the body. In: Fluid Dynamics Transactions. Vol. 3, Warszawa, PWN, 1966.

11. Ryzhov, O. S. and Terent'ev, E. D., On the general theory of almost self-similar nonstationary flows. PMM Vol. 37, № 1, 1973.

12. Ryzhov, O. S. and Terent'ev, E. D., On the hypersonic flow past a lift airfoil. PMM Vol. 38, № 1, 1974.

13. Ryzhov, O. S. and Terent'ev, E. D., On space hypersonic flows. PMM Vol. 39, № 3, 1975.

14. Sedov, L. I., Similarity and Dimensional Analysis Methods in Mechanics. Moscow, "Nauka", 1967.

15. Terent'ev, E. D., On integrals of equations of unstable nearly self-similar flows. PMM Vol. 39, № 6, 1975.

Translated by L. K.